TRINITY COLLEGE

Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures | | | | | | | | |

In words ______________________________

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Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	48	35
Section Two: Calculator-assumed	13	13	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (48 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

(a) Determine $\int \cot(2x)\,dx$. (2 marks)

Solution
$\int \cot 2x\,dx = \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\,dx$ $= \frac{1}{2}\ln\lvert \sin 2x \rvert + c$
Specific behaviours
✓ writes in form $f'(x) \div f(x)$ ✓ correct integral with constant

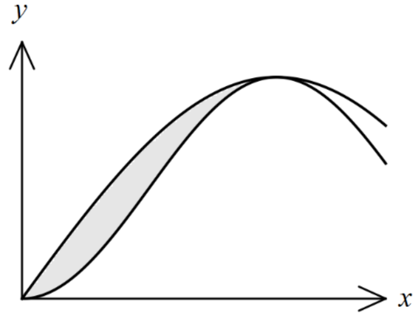
(b) Evaluate $\int_0^{\frac{\pi}{2}} \left(3 - \sec^2\left(\frac{x}{2}\right) + 3\tan^2\left(\frac{x}{2}\right)\right) dx$. (3 marks)

Solution
$\int_0^{\frac{\pi}{2}} 3 - \sec^2\left(\frac{x}{2}\right) + 3\tan^2\left(\frac{x}{2}\right) dx = \int_0^{\frac{\pi}{2}} 2\sec^2\left(\frac{x}{2}\right) dx$ $= \left[2(2)\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$ $= 4$
Specific behaviours
✓ simplifies   ✓ antiderivative   ✓ evaluates

Question 2 **(5 marks)**

The curves $y = 3\sin\left(\frac{x}{2}\right)$ and $y = 3\sin^2\left(\frac{x}{2}\right)$ are shown to the right.

Determine the area of the shaded region trapped between the curves.



Solution
$3\sin\left(\frac{x}{2}\right)\left(1-\sin\left(\frac{x}{2}\right)\right) = 0 \Rightarrow x = 0, \pi$ $A = \int_0^{\pi} 3\sin\left(\frac{x}{2}\right) - 3\sin^2\left(\frac{x}{2}\right)\,dx$ $= \int_0^{\pi} 3\sin\left(\frac{x}{2}\right) - \frac{3}{2}(1-\cos x)\,dx$ $= \left[-6\cos\left(\frac{x}{2}\right) - \frac{3x}{2} + \sin x\right]_0^{\pi}$ $= \left(0 - \frac{3\pi}{2} + 0\right) - (-6 - 0 + 0)$ $= 6 - \frac{3\pi}{2}$
Specific behaviours
✓ indicates bounds   ✓ writes required integral   ✓ uses double angle identity   ✓ antidifferentiates   ✓ substitutes and simplifies

Question 3 **(6 marks)**

Consider the system of equations (where c is a constant) given by

$$x + y + cz + 1 = 0$$
$$2x + y - z - 5 = 0$$
$$2x + 2y + z = 0$$

(a) When $c = 1$, solve the system of equations and interpret your solution geometrically. (4 marks)

Solution
Eliminate using $2(1) - (2)$: $2x + 2y + 2z = -2$; $2x + 2y + z = 0$; $z = -2$ Substitute for z and then $(2) - (1)$: $x + y = 1$; $2x + y = 3$; $x = 2$ Substitute for x, z and then(1): $2 + y - 2 + 1 = 0$; $y = -1$ Hence the system is three planes that intersect at the point $(2, -1, -2)$.
Specific behaviours
✓ eliminates a variable correctly   ✓ solves for one variable   ✓ solves for all variables   ✓ states three planes intersecting at a point

(b) State the value of c for which the system of equations has no solution and explain the geometric interpretation of this. (2 marks)

Solution
No solution when $c = \frac{1}{2}$. *[(1) and (3) have parallel normals.]*   With this value, the system represents two non-coincident parallel planes cut by a third plane.
Specific behaviours
✓ states value   ✓ geometric interpretation

Question 4 **(6 marks)**

Consider $f(z) = z^4 + kz^3 + 9z^2 - 8z + 20$, where k is a real constant.

The equation $f(z) = 0$ has a solution $z = 1 - 2i$.

(a) State a second solution to $f(z) = 0$. (1 mark)

Solution
$z = 1 + 2i$
Specific behaviours
✓ indicates conjugate root

(b) Deduce that $k = -2$. (3 marks)

Solution
$f(z) = (z - 1 + 2i)(z - 1 - 2i)(z^2 + az + b)$ $= (z^2 - 2z + 5)(z^2 + az + b)$ Consider constant term: $20 = 5b \Rightarrow b = 4$ Consider z coefficient: $-8 = -2b + 5a \Rightarrow a = 0$ Hence z^3 coefficient $k = (1)(0) + (-2)(1) \Rightarrow k = -2$.
Specific behaviours
✓ factors f into known and unknown quadratics ✓ deduces constant term b in quadratic ✓ deduces z coefficient a in quadratic and hence value of k

(c) Determine all other solutions of the equation $f(z) = 0$. (2 marks)

Solution
$z^2 + 4 = 0$ $z = \pm 2i$
Specific behaviours
✓ factors quadratic ✓ states both solutions

Question 5 **(7 marks)**

Functions f, g and h are defined by $f(x) = \sqrt{25 - x}$, $g(x) = \dfrac{10}{x}$ and $h(x) = f \circ g(x)$.

(a) Determine the defining rule for $h(x)$ and state its domain. (4 marks)

Solution
$$h(x) = f \circ g(x) = \sqrt{25 - \frac{10}{x}}$$ Note, $x \neq 0$. When $x < 0$, the radicand $\left(25 - \frac{10}{x}\right)$ will always be positive. When $x > 0$, require radicand to be positive: $$25 - \frac{10}{x} \geq 0 \Rightarrow x \geq \frac{2}{5}$$ Hence domain: $$D_h = \left\{x \in \mathbb{R}, x < 0 \cup x \geq \frac{2}{5}\right\}$$
Specific behaviours
✓ defining rule ✓ domain excludes zero ✓ domain allows all $x < 0$ ✓ domain allows all $x \geq 0.4$

(b) Determine the defining rule for $h^{-1}(x)$ and state its range. (3 marks)

Solution
Inverse: $$x^2 = 25 - \frac{10}{y}$$ $$\frac{10}{y} = 25 - x^2$$ $$y = h^{-1}(x) = \frac{10}{25 - x^2}$$ Range of inverse same as domain of function: $$R_{h^{-1}} = \left\{y \in \mathbb{R}, y < 0 \cup y \geq \frac{2}{5}\right\}$$
Specific behaviours
✓ swaps variables and eliminates root ✓ correct inverse ✓ uses domain from part (a) for range of inverse

Question 6 **(6 marks)**

(a) Express $\frac{6}{(u-3)(u+3)}$ in the form $\frac{a}{u-3}+\frac{b}{u+3}$. (2 marks)

Solution
$6=a(u+3)+b(u-3)$ $u=3\Rightarrow a=1,\qquad u=-3\Rightarrow b=-1$ $\frac{6}{(u-3)(u+3)}=\frac{1}{(u-3)}-\frac{1}{(u+3)}$
Specific behaviours
✓ indicates appropriate method ✓ correct partial fractions

(b) Use the substitution $u^2=x+7$ to determine the indefinite integral I shown below in the form $\ln(g(x))+c$ for $x>2$. (4 marks)

$$I=\int\frac{3}{(x-2)\sqrt{x+7}}dx$$

Solution
$u^2=x+7\Rightarrow 2u\,du=dx$ $I=\int\frac{3\times 2u}{(u^2-9)u}du$ $=\int\frac{6}{u^2-9}du$ $=\int\frac{1}{u-3}-\frac{1}{u+3}du$ $=\ln\left(\frac{u-3}{u+3}\right)+c$ $=\ln\left(\frac{\sqrt{x+7}-3}{\sqrt{x+7}+3}\right)+c$
Specific behaviours
✓ writes integral in terms of u ✓ simplifies using result from part (a) ✓ obtains antiderivative ✓ writes in required form

Question 7 **(6 marks)**

The functions $f(x)$ and $g(x)$ are polynomials in x of degree two and degree one respectively, and their quotient is the rational function $h(x) = \dfrac{f(x)}{g(x)}$.

The graph of $y = h(x)$ passes through the point $(0, -1)$, has vertical asymptote $x = -3$ and has roots at $x = -2$ and $x = 3$.

Determine the equation of the other asymptote of the graph of $y = h(x)$.

Solution
Using roots: $$f(x) = a(x+2)(x-3)$$ Using asymptote: $$g(x) = m(x+3)$$ Hence $$h(x) = \frac{f(x)}{g(x)} = \frac{a(x+2)(x-3)}{m(x+3)}$$ Using point: $$-1 = \frac{-6a}{3m} \Rightarrow \frac{a}{m} = \frac{1}{2}$$ Hence $$h(x) = \frac{1}{2}\left(\frac{(x+2)(x-3)}{x+3}\right) = \frac{1}{2}\left(\frac{x^2-x-6}{x+3}\right)$$ Express as proper fraction: $$h(x) = \frac{1}{2}\left(\frac{x^2+3x}{x+3} + \frac{-4x-12}{x+3} + \frac{6}{x+3}\right) = \frac{x}{2} - 2 + \frac{6}{2x+6}$$ Hence equation of oblique asymptote: $$y = \frac{x}{2} - 2$$
Specific behaviours
✓ uses roots to obtain factors of f ✓ uses vertical asymptote to obtain g ✓ uses point to obtain constant ✓ indicates correct $h(x)$ ✓ expresses $h(x)$ as proper fraction ✓ states correct equation for asymptote

Question 8 **(7 marks)**

(a) Solve the equation $z^3 = 8\operatorname{cis}\left(\frac{2\pi}{3}\right)$, where $-\pi < \arg(z) < 0$. (2 marks)

Solution
$z = 2\operatorname{cis}\left(\frac{2\pi}{9} + \frac{2n\pi}{3}\right)$ Hence $z = 2\operatorname{cis}\left(-\frac{4\pi}{9}\right)$
Specific behaviours
✓ applies De Moivre's theorem ✓ correct root

(b) $1, w$ and w^2 are the three cube roots of unity.

(i) State the value of $1 + w + w^2$ and the value of w^3. (1 mark)

Solution
Sum of roots: $1 + w + w^2 = 0$ Product of roots: $(1)(w)(w^2) = w^3 = 1$
Specific behaviours
✓ sum and product

Let $u = (1 + 3w + w^2)^2$ and $v = (1 + w + 3w^2)^2$.

(ii) Show that $u + v = -4$ and $uv = 16$. (4 marks)

Solution
$u = (1 + w + w^2 + 2w)^2$ $= (0 + 2w)^2 = 4w^2$ $v = (1 + w + w^2 + 2w^2)$ $= (0 + 2w^2)^2 = 4w^4 = 4w$ Hence $u + v = 4w^2 + 4w$ $= 4(w^2 + w + 1) - 4$ $= -4$ And $uv = 4w^2 \times 4w$ $= 16w^3$ $= 16$
Specific behaviours
✓ simplifies u ✓ simplifies v ✓ derives value for sum ✓ derives value for product

End of questions

Supplementary page

Question number: _________